Receptor ligand binding kinetics

In the simplest case: $A + B \leftrightarrow C$, the system differential equation can be reduced to:

$$dC(t)/dt = ka\ (Ao - C(t))\ (Bo - C(t)) - kd\ C(t) \quad (1)$$

Almost always, this equation (1) is resolved only in the hypothesis that C(t) remains small in front of Ao (the total concentration of receptor) which means a low receptor occupancy. This is always the case when Bo (the total concentration of ligand) is in tracer concentration.
In this case, the equation (1) becomes linear:

$$dC(t)/dt = ka\ Ao\ (Bo - C(t)) - kd\ C(t)$$

$$dC(t)/dt = ka\ Ao\ Bo - (ka\ Ao + kd)\ C(t)$$

and the solution is simple (assuming C(0) = 0):

$$C(t) = ka\ Ao\ Bo\ (1 - \exp(- (ka\ Ao + kd)\ t) / (ka\ Ao - kd)$$

Which gives:

$$C(t) = Ao\ Bo\ (1 - xp(- (ka\ Ao + kd)t) / (Ao + Kd) \quad (2)$$

However, this simplification is not necessary to derive an analytical solution because the differential equation is an equation with separate variables:

$$dC(t) / (ka\ (Ao - C(t))\ (Bo - C(t)) - kd\ C(t)) = dt$$

Factoring the denominator is performed using the solutions in the second-degree equation:

$$ka\ C(t)^2 - (ka\ (Ao + Bo) + kd)\ C(t) + ka\ Ao\ Bo = 0$$

By posing D (discriminant) = $(ka\ (Ao + Bo) - kd)^2 - 4\ ka^2\ Ao\ Bo$, we have:

$$K_{1,2} - (ka\ (Ao + Bo) + kd) \pm D^{½}) / 2\ ka$$

And (1) is written:

$$dC(t) / ka\ (C(t) - K_1)\ (C(t) - K_2) = dt$$

or:

$$dC(t)\ (1/(C(t) - K_1) - 1/(C(t) - K_2) = ka\ (K_1 - K_2)\ dt$$

which integrates in a trivial way as:

$$Ln((C(t) - K_1)/(C(t) - K_2)) - ka\ (K_1 - K_2)\ t - u$$

We notice that $ka\ (K_1 - K_2) = D^{½}$ and, when C(0) = 0, $u = Ln(K_1/K_2)$ and the solution is:

$$C(t) = K_1\ K_2\ (1 - \exp(D^{½}\ t)) / (K_2 - K_1\ \exp(D^{½}\ t)$$

or (to show the decreasing exponential:

$$C(t) = K_1\ K_2\ (1 - \exp(- D^{½}\ t) / (K_1 - K_2\ \exp(- D^{½}\ t) \quad (3)$$

We find that at the equilibrium ($t \rightarrow \infty$) $C(\infty) = K_2$ (the solution K_1 is obviously not suitable) and if we assume that we are far from saturation (K_2 small) we find the simplified solution (2).

Surprisingly, the general solution (3) is very rarely used.